TAME FUNCTIONS WITH STRONGLY ISOLATED SINGULARITIES AT INFINITY: A TAME VERSION OF A PARUSIŃSKI'S THEOREM

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Abstract. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a definable function, enough differentiable. Under the condition of having strongly isolated singularities at infinity at a regular value c, we give a sufficient condition expressed in terms of the total absolute curvature function to ensure the local triviality of f over a neighbourhood of c and doing so providing the tame version of Parusiński's Theorem on complex polynomials with isolated singularities at infinity.

1. Introduction

The fundamental result of Thom about the finiteness of the topological types of a given polynomial function [Th], has led to some understanding of the geometry of the foliation by the level of a given tame function $f : U \subset \mathbb{K}^n \mapsto \mathbb{K}$ nearby a generalised critical value, value likely to be a bifurcation value, that is at which the topology of the fibres is not locally constant. Rather early one noticed that a bifurcation value could be a regular value, as already suggested by the properness condition in Erehsmann's Theorem to ensure the local triviality of a submersion. For a decade or so, there were no "effective" criterion to describe these regular bifurcation values, or at least a finite subset of $\mathbb{K}$ that would contain them. Then came some sufficient conditions to trivialise the given function over a regular value (see [Ph], [Br], [HL]). This led rapidly to the notion of asymptotic critical value (or generalised critical value), requiring, similarly to the vanishing of the gradient at a critical point, that the gradient vector field is asymptotically small along a sequence going to the boundary of the domain. To be more precise there exists a sequence x going to "infinity" along which $f(x) \to c \in \mathbb{K}$ and $|x| \cdot |\nabla f(x)| \to 0$.

For a tame function defined on $\mathbb{K}^n$ the boundary must be understood as infinity, more precisely the hyperplane at infinity of the usual projective compactification of $\mathbb{K}^n$. In this context, a regular bifurcation value seems to be like a critical value coming from some sort of singular phenomenon (to be fully understood) lying on the boundary. Then it was proved ([Ph], [HL],

2000 *Mathematics Subject Classification.* 58K05, 03C64, 14B05.

Key words and phrases. generalised critical values, bifurcation values, o-minimal structure, relative conormal space, total absolute curvature, polar curves.

[Pa1], [Ti], [LZ], [D'A1], etc...) that any regular bifurcation value must be an asymptotic critical value and there are finitely many such asymptotic critical values.

Hà and Lê proved in [HL] that the triviality of the complex plane polynomial function f over a neighbourhood of a given value c was equivalent to the constancy of the Euler Characteristic of the fibres in a neighbourhood of the value c. This result was later generalised by Parusiński to the case of complex polynomials with isolated singularities at infinity [Pa1], which he also proved to be equivalent to requiring that the regular value c is not an asymptotic critical value. Parusiński [Pa2] also explained that being an asymptotic critical value of f is equivalent to the failure of a certain stratifying condition on the projective closure of the graph of f, expressed as a property of the relative conormal space of f, what Tibăr has called t-isolated singularities ([Ti]). Despite the same property holds true for real polynomials ([Ti]), the phenomena occurring in the real domain are of a much less rigid kind. There are already counter-example of Hà-Lê's result as noticed by Tibăr and Zaharia [TZ]. They nevertheless provided necessary and sufficient conditions for a real plane polynomial function to be locally trivial over a neighbourhood of a value c. Later, in [CP], Coste and de la Puente proved an equivalent version of Tibăr-Zaharia's result in terms of polar curves, that is involving the relative conormal geometry of the function at infinity nearby the given level c. In the world of real tame functions (to be understood as "globally" definable in an o-minimal structure expanding the ordered field of real numbers), the hope to find necessary and sufficient conditions for a regular value to be a bifurcation value is much harder to hold ! Different sufficient conditions were provided to guarantee the local triviality at infinity over a neighbourhood of a regular value (see [LZ], [Ti], [TZ], [D'A1], [DG1], [DG2], [DG3]). There are unfortunately not all comparable, but they all exhale a similar flavour: The naive belief that too much bending (curvature and so a possible lack of transversality to "spheres") is an obstruction to trivialisation. Thus the understanding of the relative conormal geometry at infinity nearby a regular level c we are interested in is an important aspect to explore in order to deciding whether the value c is a bifurcation value or not.

The aim of this paper is to provide a real version in the (globally) definable setting of Parusiński's result.

We propose a condition on the fibre, at a regular value c, of the relative conormal space of a tame function f, that we call **SISI** at c (shortening for strongly isolated singularities at infinity, see Definition 6.1). This condition geometrically means: For a given value c, there are at most finitely many points at infinity such that any limit of tangent hyperplanes, along any sequence going to infinity with limit of secants the given point at infinity and along which the function tends to c, may not be orthogonal to the line direction corresponding to this point.

This property, about the asymptotic behaviour of limits of tangent hyperplanes to the fibres of f when getting closer and closer to the level c in a neighbourhood of infinity, when combined with a property on the total absolute curvature of the function f, ensures the triviality of f nearby c (Theorem 6.2). This condition is satisfied for a real polynomial having isolated singularities at infinity as defined by Parusiński. We have stated our result in terms of the continuity of the total absolute curvature function of f which is the same sort of condition of having the generic polar curves empty at infinity in a neighbourhood of the level c.

The paper is organised as follows:

We begin with Section 2 in which we explain some notations and some conventions that will be later used in the paper.

Section 3, 4 and 5 are reminders of the definitions and of some of the elementary properties of the key objects we work with, such as the total absolute curvature function, the relative conormal space and the notion of asymptotic critical value, we want to focus on in Sections 6 and 7.

Condition **SISI** is defined in Section 6, where we state and proof our main result:

Theorem 6.2. *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a C^l definable function with $l \geqslant 2$. Assume that the function f satisfies condition* **SISI** *at a regular value c. If the function $t \mapsto |K|(t)$ is continuous at c, we trivialise f over a neighbourhood of c by means of the flow of a C^{l-1} vector field. So $c \notin B(f)$.*

In Section 7, we compare condition **SISI** for a real polynomial function and Parusiński's notion of *isolated singularities at infinity* (defined for complex polynomials). Our other main result is

Proposition 7.4. *If $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a real polynomial with isolated singularities at infinity, then condition* **SISI** *at any regular value c is satisfied.*

We finish with some final remarks and comments in Section 8 to explain that our result is really Parusiński real counter-part and, unfortunately, nothing much better is to be expected for real polynomials that what we do not already have with this level of generality.

2. Notation - convention

Let $\mathbb{R}^n$ be the real n-dimensional affine space endowed with its Euclidean metric. The scalar product will be denoted by $\langle \cdot, \cdot \rangle$.

Let $\mathbb{P}^n_{\mathbb{K}}$ be the projectivised space of the $\mathbb{K}$-vector space $\mathbb{K}^n$. This notation will be exclusively used to mean "the projective" compactification of $\mathbb{K}^n$, where $\mathbb{K}$ either stands for $\mathbb{R}$ or $\mathbb{C}$.

Let $\mathbf{B}^n_R$ be the open ball of $\mathbb{R}^n$ centred at the origin and of radius $R > 0$.

Let $\mathbf{S}^{n-1}_R$ be the $(n-1)$-sphere centred at the origin and of radius $R > 0$.

Let $\mathbf{S}^{n-1}$ be unit ball of $\mathbb{R}^n$.

Let $\mathbf{h}$ be a $\mathbb{K}$-vector subspace of dimension q of a $\mathbb{K}$-vector space E. Let $\mathbf{G}_{\mathbb{K}}(p, \mathbf{h})$ be the Grassmann manifold of the p-dimensional $\mathbb{K}$-vector subspaces of $\mathbf{h}$. When $\mathbf{h} = E = \mathbb{K}^q$, we will only write $\mathbf{G}_{\mathbb{K}}(p, q)$.

Let us recall briefly what an o-minimal structure is.

An *o-minimal structure* $\mathcal{M}$ *expanding the ordered field of real numbers* is a collection $(\mathcal{M}_p)_{p\in\mathbb{N}}$, where $\mathcal{M}_p$ is a set of subsets of $\mathbb{R}^p$ satisfying the following axioms

1) For each $p \in \mathbb{N}$, $\mathcal{M}_p$ is a boolean subalgebra of subsets of $\mathbb{R}^p$.
2) If $A \in \mathcal{M}_p$ and $B \in \mathcal{M}_q$, then $A \times B \in \mathcal{M}_{p+q}$.
3) If $\pi : \mathbb{R}^{p+1} \mapsto \mathbb{R}^p$, is the projection on the first p factors, given any $A \in \mathcal{M}_{p+1}$, $\pi(A) \in \mathcal{M}_p$.
4) The algebraic subsets of $\mathbb{R}^p$ belongs to $\mathcal{M}_p$.
5) $\mathcal{M}_1$ consists exactly of the finite unions of points and intervals.

So the smallest o-minimal structure is the structure of the semi-algebraic subsets.

Assume that such an o-minimal structure $\mathcal{M}$ is given for the rest of this article.

A subset A of $\mathbb{R}^p$ is a *definable subset* (in the given o-minimal structure) of $\mathbb{R}^p$, if $A \in \mathcal{M}_p$.

A subset B of $\mathbb{P}^p_{\mathbb{R}}$ is said to be *(globally) definable* if its trace in each affine chart is a definable subset of this chart.

For mappings we slightly restrict the usual definition of a definable mapping. A mapping $g : X \mapsto Y$, where $X \subset \mathbb{P}^p_R$ and $Y \subset \mathbb{R}^q$, is a *definable mapping* (or just definable, for short) if the intersection of the closure of its graph in $\mathbb{P}^p_{\mathbb{R}} \times \mathbb{R}^q$ with $\mathbb{P}^p_{\mathbb{R}} \times \mathbf{B}$ is a definable subset of $\mathbb{P}^p_{\mathbb{R}} \times \mathbb{R}^q$ for any ball $\mathbf{B} \subset \mathbb{R}^q$.

The reader may refer to [Co, vD, vDM] to learn more about the properties of definable subsets and definable mappings.

Let S be a C^1 definable submanifold of $\mathbb{P}^n_{\mathbb{R}}$. Let $g : S \mapsto \mathbb{R}^q$ be a C^1 definable mapping. The critical set of g is denoted by $\mathbf{crit}(g)$.

By abuse of language, we will talk about the rank of the mapping g at a point x_0 to mean the rank of the differential $\mathrm{d}_{x_0} g$. We will also talk about the rank of g to mean the maximal rank of the differentials $\mathrm{d}_x g$, $x \in S$.

Remark 2.1. *In this paper, we will* always *use the adjective* definable *for a subset of an affine space to mean* definable *in the projective compactification of the ambient affine space.*

Let φ and ψ be two germs at the origin (resp. at infinity) of single real variable functions. We write $\varphi \sim \psi$ to mean that the ratio φ/ψ has a non zero finite limit at the origin (resp. at infinity). We write $\varphi \simeq \psi$ when the limit of φ/ψ at the origin (resp. at infinity) is 1. We will write $\psi = o(\varphi)$ to mean $\psi/\varphi \to 0$ at the origin (resp. at infinity).

3. Gauss Map of a tame function

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a C^l definable function, with $l \geqslant 2$.

Let $\mathbf{crit}(f)$ be the critical set of the function f and let $K_0(f)$ be the set of its critical values, that is $K_0(f) = f(\mathbf{crit}(f))$, that we recall is finite.

For each t, let F_t be the level $f^{-1}(t)$.

The Gauss map of the function f is the mapping defined as follows:

$$\begin{array}{rccc} \nu_f & : & \mathbb{R}^n \setminus \mathbf{crit}(f) & \mapsto & \mathbf{S}^{n-1} \\ & & x & \mapsto & \dfrac{\nabla f(x)}{|\nabla f(x)|} \end{array}$$

It is a definable mapping that is C^{l-1}. Thus the set of its critical values $\nu_f(\mathbf{crit}(\nu_f))$ is a definable subset of dimension at most $n-2$.

For each regular value t, let ν_t be the restriction of ν_f to F_t. So it is also a Gauss map on each connected components of F_t providing each component with an orientation that is compatible with the transverse structure of the foliation of $\mathbb{R}^n \setminus \mathbf{crit}(f)$ by the levels of the function f. Note also that $\mathbf{crit}(\nu_f) \cap F_t = \mathbf{crit}(\nu_t)$.

For a given $x \in F_t$, let $k_t(x)$ be the Gaussian curvature of F_t at x, namely $k_t(x) = \det(\mathrm{d}_x \nu_t)$.

Let $\mathrm{d}v_{n-1}$ be the $(n-1)$-dimensional Hausdorff measure of $\mathbb{R}^n$.

Definition 3.1. *Let t be a regular value of the function f.*

(i) The total absolute curvature of the level F_t is

$$|K|(t) = \int_{F_t} |k_t(x)| \mathrm{d}v_{n-1}(x)$$

(ii) The total curvature of the level F_t is

$$K(t) = \int_{F_t} k_t(x) \mathrm{d}v_{n-1}(x)$$

Let us say few words about these total curvatures. First they are well defined as it will appeared below.

Let us define $\Psi_f : \mathbb{R}^n \setminus (\mathbf{crit}(f) \cup \mathbf{crit}(\nu_f)) \mapsto \mathbf{S}^{n-1} \times \mathbb{R}$ such that $x \mapsto \Psi_f(x) := (\nu_f(x), f(x))$.

Let $\widetilde{\mathcal{U}}$ be the image of Ψ_f. It is an open definable subset since Ψ_f is a local diffeomorphism at each of its point. The subset $\widetilde{\mathcal{U}}$ is a finite disjoint union $\sqcup_m \widetilde{\mathcal{U}}_m$, where $\widetilde{\mathcal{U}}_m = \{(u,t) : \#\Psi_f^{-1}(u,t) = m\}$. For any regular value t, let $\mathcal{U}_t := \{u \in \mathbf{S}^{n-1} : (u,t) :\in \widetilde{\mathcal{U}}\}$. It is an open definable subset of $\mathbf{S}^{n-1}$. Let $(\mathcal{U}_{i,t})_{i=1,\ldots,q_t}$ be the set of connected components of $\mathcal{U}_t$. For each $i = 1, \ldots, q_t$, let $m_{i,t}$ be $\#\Psi_f^{-1}(u,t)$, for any $u \in \mathcal{U}_{i,t}$.

From Gabrielov's uniformity principle there exists a positive integer N_f such that for each $(u,t) \in \widetilde{\mathcal{U}}$, we deduce $\#(\nu_t^{-1}(u) \cap (F_t \setminus \mathbf{crit}(\nu_f))) \leqslant N_f$. So we find

$$|K|(t) = \sum_{i=1}^{q_t} m_{i,t}\mathrm{vol}_{n-1}(\mathcal{U}_{i,t}), \text{ and } K(t) = \sum_{i=1}^{q_t} \delta_{i,t}\mathrm{vol}_{n-1}(\mathcal{U}^t_{m,i}),$$

where $\delta_{i,t}$ is the degree of the mapping ν_t at any $x \in \nu_t^{-1}(u)$, $u \in \mathcal{U}_{i,t}$.

Thus we have defined two functions $|K| : \mathbb{R} \setminus K_0(f) \mapsto \mathbb{R}$, $t \mapsto |K|(t)$, the total absolute curvature function and $K : \mathbb{R} \setminus K_0(f) \mapsto \mathbb{R}$, $t \mapsto K(t)$, the total curvature function.

It is a matter of interest to know more about the regularity properties of these two functions, since they are closely linked to the topology of the levels F_t, as the usual Gauss-Bonnet-Chern Theorem suggests in the compact connected odd-dimensional case.

In this general setting little is known about such functions, nevertheless we know that

Theorem 3.2 ([Gr]). *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a C^l definable function, with $l \geqslant 2$.*
(i) The function $t \mapsto |K|(t)$ has at most finitely many discontinuities.
(ii) If the function $t \mapsto |K|(t)$ is continuous at a regular value c, so is $t \mapsto K(t)$.

Obviously if the Gauss map ν_f is degenerate, that is of rank at most $n-2$, the former results are without interest since both total curvature functions are the null function.

Since there are finitely many values at which $|K|$ may not be continuous and finitely many values at which the topology of the fibres of f is not locally constant, is there a link between these two set of values ?
We will see in the next sections that with additional hypotheses there are such relations.

To finish this section let us state the following result that will be important in Section 6.

Proposition 3.3 ([Gr, Corollary 6.3]). *Let c be a regular value at which the function $|K|$ is not continuous. There exists an open subset $U \subset \mathbf{S}^{n-1}$, such that for any $u \in U$, there exists a connected component Γ of $\Psi_f^{-1}(\{u\} \times \mathbb{R})$, such that $\Gamma \cap f^{-1}(c)$ is not empty and one of the two situations below happens:*
(i) If c is the infimum of f along Γ, for any $\varepsilon > 0$ small enough, $\Gamma \cap f^{-1}(]c, c+\varepsilon[)$ is not bounded.
(ii) If c is the supremum of f along Γ, for any $\varepsilon > 0$ small enough, $\Gamma \cap f^{-1}(]c-\varepsilon, c[)$ is not bounded.

Note that obviously the oriented polar curve $\Psi_f^{-1}(\{u\} \times \mathbb{R})$ is C^{l-1} and has at most N_f connected components lying in $\mathbb{R}^n \setminus (\mathbf{crit}(f) \cup \mathbf{crit}(\nu_f))$.

4. Relative conormal geometry at infinity of a tame function

Let $\mathbb{H}^\infty_{\mathbb{K}} := \mathbb{P}^n_{\mathbb{K}} \setminus \mathbb{K}^n$ be the *hyperplane at infinity.*

Definition 4.1. *Let $g : S \mapsto \mathbb{R}$ be a C^l definable mapping from a submanifold $S \subset \mathbb{R}^n$. The relative projective conormal bundle of the function g is the subset $\mathscr{X}_g$ of $\mathbb{P}^n_{\mathbb{R}} \times \mathbf{G}_{\mathbb{R}}(n-1, n) \times \mathbb{R}$ defined as the closure of*

$$\{(x, \mathbf{h}, t) \in (S \setminus \mathbf{crit}(g)) \times \mathbf{G}_{\mathbb{R}}(n-1, n) \times \mathbb{R} : T_x g \subset \mathbf{h}, t = g(x)\},$$

where $T_x g = T_x(g^{-1}(g(x)))$.

The subset $\mathscr{X}_g$ is a closed definable subset of $\mathbb{P}^n_{\mathbb{R}} \times \mathbf{G}_{\mathbb{R}}(n-1, n) \times \mathbb{R}$ of dimension n. Note that $\mathscr{X}_g \cap ((S \setminus \mathbf{crit}(g)) \times \mathbf{G}_{\mathbb{R}}(n-1, n) \times \mathbb{R})$ is a C^{l-1} submanifold of $\mathbb{P}^n_{\mathbb{R}} \times \mathbf{G}_{\mathbb{R}}(n-1, n) \times \mathbb{R}$ since the $x \mapsto T_x g \in \mathbf{G}_{\mathbb{R}}(\dim S, n)$ is just the projective Gauss map.

Let $(\pi_g, \tau_g, t_g) : \mathscr{X}_g \mapsto \mathbb{P}^n_{\mathbb{R}} \times \mathbf{G}_{\mathbb{R}}(n-1, n) \times \mathbb{R}$ be the restriction of projections on the respective factors of $\mathbb{P}^n_{\mathbb{R}} \times \mathbf{G}_{\mathbb{R}}(n-1, n) \times \mathbb{R}$, that is,

$$\pi_g(x, t, \mathbf{h}) = x, \ \tau_g(x, t, \mathbf{h}) = \mathbf{h} \text{ and } t_g(x, t, \mathbf{h}) = t.$$

Those maps are definable, and C^{l-1} on $\mathscr{X}_g \cap ((S \setminus \mathbf{crit}(g)) \times \mathbf{G}_{\mathbb{R}}(n-1, n) \times \mathbb{R})$.

Remark 4.2. *The space* $(\pi_g, \tau_g)(\mathscr{X}_f)$ *is also known as* the relative conormal space *of the function* f.

Let f be as in section 3.

Since $\dim \mathscr{X}_f \cap (\mathbb{R}^n \times \mathbb{R} \times \mathbf{G}_{\mathbb{R}}(n-1, n)) = n$, defining $\mathscr{X}_f^\infty$ and $\mathscr{X}_t^\infty$ respectively, for a regular value t, as

$$\mathscr{X}_f^\infty := \mathscr{X}_f \cap (\mathbb{H}^\infty_{\mathbb{R}} \times \mathbf{G}_{\mathbb{R}}(n-1, n) \times \mathbb{R}) \text{ and}$$
$$\mathscr{X}_t^\infty = \mathscr{X}_f \cap (\mathbb{H}^\infty_{\mathbb{R}} \times \mathbf{G}_{\mathbb{R}}(n-1, n) \times \{t\}),$$

we deduce $\dim \mathscr{X}_f^\infty \leqslant n-1$ and $\dim \mathscr{X}_t^\infty \leqslant n-1$.

Let $X_f = (\pi_f, t_f)(\mathscr{X}_f)$, then X_f is definable and $\dim X_f = n$. Note that X_f is the projective closure of the graph of the function f.

Let $X_f^\infty = (\pi_f, t_f)(\mathscr{X}_f^\infty) = X_f \cap (\mathbb{H}^\infty_{\mathbb{R}} \times \mathbb{R})$, thus X_f^∞ is definable and $\dim X_f^\infty \leqslant n-1$.

If $(\lambda, t) \in X_f^\infty$, let $\Omega_{\lambda,t} \subset \mathbf{G}_{\mathbb{R}}(n-1, n)$ be $\tau_f((\pi_f, t_f)^{-1}(\lambda, t))$.

Remark 4.3. *If the subset of limits of tangent hyperplanes* $\Omega_{\lambda,t}$ *is finite, from section 3, we deduce* $\#\Omega_{\lambda,t} \leqslant 2N_f$.

We finally define $X_t^\infty = \pi_f(\mathscr{X}_t^\infty) \subset \mathbb{H}^\infty_{\mathbb{R}}$, which is definable and of dimension at most $n-1$. It is important to note that it may strictly contain $\mathbb{H}^\infty_{\mathbb{R}} \cap \mathbf{clos}(F_t)$, where $\mathbf{clos}(F_t) \subset \mathbb{P}^n_{\mathbb{R}}$ is the projective closure of the level F_t.

As a consequence of these definitions we get the following

Corollary 4.4. *(i) There exist at most finitely many values* $t \in \mathbb{R}$ *such that* $\mathscr{X}_t^\infty$ *is of dimension exactly* $n-1$.

(ii) There exist at most finitely many points $(\lambda, t) \in X_f^\infty$ *such that* $\Omega_{\lambda,t}$ *is of dimension* $n-1$.

Proof. Since $(X_t^\infty)_{t \in \mathbb{R}}$ is a definable family of subsets whose union is X_f^∞ we then get the first point.

The second point is true for exactly the same reason for the definable family $(\Omega_{\lambda,t})_{\{(\lambda,t)\}}$. ■

First, if $\Omega_{\lambda,t}$ is of dimension $n-1$ so is $\mathscr{X}_t^\infty$. Once more what can be said about these values at which $\mathscr{X}_t^\infty$ is of dimension exactly $n-1$ or at which $\Omega_{\lambda,t}$ is of dimension $n-1$?

5. Asymptotic critical values and bifurcation values

In this section we will deal with definable functions as well as with complex polynomials. In the complex domain we will understand differentiability in the complex meaning.

Let us begin with the following

Theorem 5.1 ([Th],[Ve],[Ha],...). *Let $f : \mathbb{K}^n \mapsto \mathbb{K}$ be either a C^l definable function or a complex polynomial. There exists a smallest finite subset $B(f)$ of $\mathbb{K}$, called the set of bifurcation values of the function g such that for each $c \notin B(f)$ there exists an open neighbourhood D of c that does not meet with $B(f)$ and such that $f_{|D}$ induces a C^{l-1} trivial fibration over D, that is $f_{|D}^{-1}(D)$ is C^{l-1}-diffeomorphic to $D \times f^{-1}(c)$.*

Obviously the critical values are bifurcations values (if you do not weaken the trivialisation to be only continuous as with the real function $f(t) = t^3$). But unfortunately, there may also exist regular values through which the topology of the fibres is changing. In his original work using stratification theory, Thom did not provide any means to recognise which regular value is likely to be a bifurcation value !

For g a C^1 function $\mathbb{K}^n \mapsto \mathbb{K}$, let ∇g be the vector field $\sum_i \partial_{x_i} g \frac{\partial}{\partial x_i}$.
Let us recall what the Malgrange condition is.

Definition 5.2. *Let $g : \mathbb{K}^n \mapsto \mathbb{K}$, be a C^1 function. The function g satisfies the* Malgrange condition *at $c \in \mathbb{K}$, if for each $R \gg 1$, there exist positive constants C and η such that*

$$x \in \{y \in \mathbb{K}^n : |y| > R, |f(y) - c| < \eta\} \Longrightarrow |x| \cdot |\nabla g(x)| > C.$$

Definition 5.3. *A value $c \in \mathbb{K}$ is an* asymptotic critical value *of the C^1 function $g : \mathbb{K}^n \mapsto \mathbb{K}$, if the Malgrange condition is not satisfied at c, that is there exists a sequence (x) of points of $\mathbb{K}^n$ such that*
(i) $|x| \to +\infty$,
(ii) $g(x) \to c$ and,
(iii) $|x| \cdot |\nabla g(x)| \to 0$ when $|x|$ goes to $+\infty$.

Let us denote by $K_\infty(g)$ the set of asymptotic critical values of the function g, and we define $K(g)$ to be $K_0(g) \cup K_\infty(g)$, the set of *generalised critical values*. Let us recall the following

Theorem 5.4 ([Ph],[HL],[Pa1],[Ti],[LZ],[D'A1],...)**.** *Let $f : \mathbb{K}^n \mapsto \mathbb{K}$ be either a C^1 definable function or a complex polynomial.*
(1) $K(f)$ is finite.
(2) $B(f) \subset K(f)$.

Let f as in Section 3. We also have

Theorem 5.5 ([DG3])**.** *Let c be a value. There exists a continuous definable function germ $\theta_c :]0, \varepsilon_c[\mapsto]0, +\infty[$ such that*
(i) there exists a constant $A > 0$ such that $\forall t \in]0, \varepsilon_c[$, $\theta_c(t) \geqslant At$.
(ii) $|x| \gg 1$ and $|f(x) - c| \ll 1 \Longrightarrow |x| \cdot |\nabla f(x)| \geqslant \theta_c(|f(x) - c|)$.
(iii) If ϕ is any definable function satisfying properties (i) and (ii), and distinct from θ_c, then there exists $0 < \varepsilon < \varepsilon_c$ such that $0 < \phi(t) < \theta_c(t)$ for $t \in]0, \varepsilon[$.

Condition (iii) also implies that, given any constant $M > 1$, there exists a sequence (x) going to infinity, along which $f(x)$ tends to c and such that $|x| \cdot |\nabla f(x)| \leqslant M\theta_c(|f(x) - c|)$. Thus c is an asymptotic critical value if and only if $\theta_c(t) \to 0$ when $t \to 0$.

Once an orthonormal system of coordinates is given, which we assume, the gradient vector field ∇f splits into two orthogonal components, namely its radial part $\partial_r f$ and its spherical part $\nabla' f$, that is

$$\text{for } x \neq 0 \ |x|\partial_r f(x) = \langle \nabla f, x \rangle \text{ and } \nabla' f = \nabla f - \partial_r f.$$

Given $a < b$ two values, let $\upsilon_{a,b} :]0, +\infty[\mapsto [0, +\infty[$ be the definable function defined as

$$\upsilon_{a,b}(r) = \max \left\{ \frac{|\partial_r f|}{|\nabla f|} : x \in f^{-1}([a, b]) \cap \mathbf{S}_r^{n-1} \right\}.$$

Let υ_c be defined as $\liminf_{\varepsilon \to 0} \upsilon_{c-\varepsilon, c+\varepsilon}$. It is again a definable function.

Let us consider values a, b such that $a < b$. Let x_0 be a point in $f^{-1}(a)$ and let γ_{x_0} be the trajectory of the vector field $\frac{\nabla f}{|\nabla f|^2}$ through x_0 such that $\gamma_{x_0}(a) = x_0$. Thus we find

$$\begin{aligned} ||\gamma_{x_0}(t)| - |x_0|| = \left| \int_a^t \frac{\mathrm{d}|\gamma_{x_0}(\tau)|}{\mathrm{d}\tau} \mathrm{d}\tau \right| &\leqslant \int_a^t \left| \frac{\mathrm{d}|\gamma_{x_0}(\tau)|}{\mathrm{d}\tau} \right| \mathrm{d}\tau \\ &\leqslant \int_a^t \left| \left\langle \frac{\mathrm{d}\gamma_{x_0}}{\mathrm{d}\tau}, \frac{\gamma_{x_0}(\tau)}{|\gamma_{x_0}(\tau)|} \right\rangle \right| \mathrm{d}\tau \\ &\leqslant \int_a^t \frac{|\partial_r f(\gamma_{x_0}(\tau))|}{|\nabla f(\gamma_{x_0})|^2} \mathrm{d}\tau . \end{aligned}$$

Lemma 5.6. *If $[a, b] \cap K(f) = \emptyset$, then $\upsilon_{a,b}(r) \to 0$ when $r \to +\infty$.*

Proof. There exists a positive constant C such that for R large enough, $|x| \cdot |\nabla f| \geqslant C$ once $x \in f^{-1}([a, b]) \setminus \mathbf{B}_R^n$.

Assume there is a positive constant A such that $v_{a,b}(r) > 2A$ for r large enough. We assume this is occurring along a definable path $\alpha : [-\varepsilon, 0] \mapsto f^{-1}([a,b]) \setminus \mathbf{B}^n_R$, such that $f \circ \alpha(s) = s + b$ and $|\alpha(s)| \to +\infty$ as s goes to 0 and verifying $v_{a,b}(\alpha(s)) \geqslant A$.
Taking the derivative respectively to s gives $\langle \nabla f(\alpha), \alpha' \rangle = 1$. Since

$$\left\langle \frac{\alpha'(s)}{|\alpha'(s)|}, \frac{\alpha(s)}{|\alpha(s)|} \right\rangle \to 1 \text{ as } s \to +\infty,$$

we deduce $|\partial_r f| \cdot |\alpha'| \to 1$, thus

$$\begin{array}{rclclcl} |\alpha'| & \leqslant & \dfrac{2}{|\partial_r f|} & \leqslant & \dfrac{2}{A|\nabla f|} & \leqslant & \dfrac{2|\alpha(s)|}{AC} \\ \dfrac{|\alpha'(s)|}{|\alpha(s)|} & \leqslant & \dfrac{2}{AC} & & & & \\ \ln(|\alpha(s)|) & \leqslant & \dfrac{2}{AC}\displaystyle\int_{-\varepsilon}^{s} \mathrm{d}t & + & \ln(|\alpha(-\varepsilon)|) & & \\ |\alpha(s)| & \leqslant & D, & & & & \end{array}$$

with $D > 0$ independent of s, which is a contradiction to $|\alpha(s)| \to +\infty$. ■

Lemma 5.7. *If $[a,b] \cap K(f) = \{b\}$, where b is a regular value such that $\theta_b^{-1}(t)$ is integrable when $t \to 0$, then $v_{a,b}(r) \to 0$ when $r \to +\infty$.*

Proof. Assume there is a positive constant A such that $v_{a,b}(r) > 2A$ for r large enough. We can assume this phenomenon is occurring along a definable path $\alpha : [-\varepsilon, 0[\mapsto f^{-1}([a,b])$, such that $f \circ (\alpha(s)) = s + b$ and $|\alpha(s)| \to +\infty$ as s goes to 0 and verifying $v_{a,b}(\alpha(s)) \geqslant A$. We also assume that ε is such that for each $s > -\varepsilon$, $|\alpha(s)|$ satisfies the point (ii) of Theorem 5.5 at b.
Taking the derivative respectively to s gives $\langle \nabla f(\alpha), \alpha' \rangle = 1$. Since

$$\left\langle \frac{\alpha'(s)}{|\alpha'(s)|}, \frac{\alpha(s)}{|\alpha(s)|} \right\rangle \to 1 \text{ as } s \to +\infty,$$

we deduce $|\partial_r f| \cdot |\alpha'| \to 1$, thus

$$\begin{array}{rclclcl} |\alpha'| & \leqslant & \dfrac{2}{|\partial_r f|} & \leqslant & \dfrac{2}{A|\nabla f|} & \leqslant & \dfrac{B|\alpha(s)|}{|\theta_b(-s)|} \text{ for a constant } B > 0 \\ \dfrac{|\alpha'(s)|}{|\alpha(s)|} & \leqslant & \dfrac{B}{|\theta_b(-s)|} & & & & \\ \ln(|\alpha(s)|) & \leqslant & B\displaystyle\int_{-\varepsilon}^{s} \frac{1}{|\theta_b(-t)|}\mathrm{d}t + \text{const.} & & & & \\ |\alpha(s)| & \leqslant & C, \text{ with } C > 0 \text{ independent of } s, & & & & \end{array}$$

which is a contradiction to $|\alpha(s)| \to +\infty$. ■

So from this we recover the following

Proposition 5.8 ([DG3])**.** *Let c be a regular value. If θ_c^{-1} is integrable nearby 0, then $c \notin B(f)$ and the trivialisation is realised by the local flow of $|\nabla f|^{-2} \cdot \nabla f$.*

When f is a real or complex polynomial (or more generally definable in a polynomially bounded o-minimal structure), given any value c there exist a smallest real number ρ_c (belonging to the field of exponents of the o-minimal structure) and a positive constant L_c such that

$$\text{if } |x| \gg 1 \text{ and } |f(x) - c| \ll 1 \text{ then } |x| \cdot |\nabla f(x)| \geqslant L_c |f(x) - c|^{\rho_c},$$

that is

$$\theta_c(t) = L_c t^{\rho_c} \text{ and thus } \rho_c \leqslant 1,$$

from [DG2] and [DG3]. Thus c is an asymptotic critical value if and only if $\rho_c > 0$. So, the integrability of θ_c is equivalent to $\rho_c < 1$. Note that also that $\rho_c < 1$ if and only if $\upsilon_c(r) \to 0$ as $r \to +\infty$. If c is a regular bifurcation value then $\rho_c = 1$.

Let us mention another result of the same kind

Theorem 5.9 ([LZ])**.** *Let c be a regular value such that $\upsilon_c < 1$. Then $c \notin B(f)$.*

See also [NZ] for the complex polynomial version of this result.

The first remark is that the trivialisation is provided by a vector field tangent to the spheres that are transverse to the levels of f nearby c near infinity. Loi and Zaharia gave a more general version: They require transversality to the levels of a proper definable positive C^1 submersion μ on $\mathbb{R}^n \setminus \{0\}$ [LZ]. They proved that $B(f) \setminus K_0(f) \subset S_\mu(f) \subset K_\infty(f)$, where $S_\mu(f)$ is defined as

$$S_\mu(f) := \left\{ b \in \mathbb{R} : \exists (x) : |x| \to +\infty, f(x) \to b, \left\langle \nu_f(x), \frac{\nabla \mu(x)}{|\nabla \mu(x)|} \right\rangle = \pm 1 \right\}.$$

To finish this section let us relate with Section 4.

Proposition 5.10. *Let c be a regular value of f such that there is $\lambda \in \mathbb{H}_{\mathbb{R}}^{\infty}$ such that $\Omega_{\lambda,c}$ is of dimension $n-1$. Then $c \in K_\infty(f)$.*

Proof. Let ξ_λ be a unit vector collinear to the line direction λ. Thus there exists an open subset $\Omega \subset \mathbf{S}^{n-1}$ such that for any $u \in \Omega$, there exists a connected component Γ_u of the oriented polar curve $\Psi_f^{-1}(\{u\} \times \mathbb{R})$ such that Γ_u is unbounded, and $f_{|\Gamma_u}(x) \to c$ as $\Gamma_u \ni x \to \infty$ such that $x/|x| \to \pm \xi_\lambda$. Thus there is a unit vector u such that $\langle u, \xi_\lambda \rangle \neq 0$. Thus for r large enough, $2\upsilon_c(r) \geqslant |\langle u, \xi_\lambda \rangle|$. ∎

The converse of this result is not true in the real context [DG2, example 5.3]. In Section 7 we will see that in the complex polynomial case, this point deserves to be discussed.

6. Triviality at infinity of tame functions with Strongly Isolated Singularity at Infinity

We still assume that f is as in Section 3. For our purpose here, we do the extra assumption that $K_0(f) = \emptyset$ and $K(f) \subset \{c\}$, since we will only deal with asymptotic critical values that are regular values.

In this section we give a sufficient condition to trivialise the function f in a neighbourhood of the regular value c that will be expressed in terms of the total absolute curvature.

We recall from Section 4 that $\mathbb{H}_{\mathbb{R}}^{\infty} = \mathbb{P}_{\mathbb{R}}^{n} \setminus \mathbb{R}^n$, so we also consider any subset of $\mathbf{G}_{\mathbb{R}}(1,n)$ as a subset of $\mathbb{H}_{\mathbb{R}}^{\infty}$ if needed.

Let us introduce the sufficient condition we just mentioned above, and that we have called the **SISI** condition.

Definition 6.1. *Let f as above. Let c be a regular value taken by f. The function f is said to have* strongly isolated singularities at infinity *at c if the following condition is satisfied:*
There exists a finite subset $\Lambda_c \subset \mathbf{G}_{\mathbb{R}}(1,n)$ such that for each line direction $\lambda \in X_c^{\infty} \setminus \Lambda_c$, for each hyperplane direction $\mathbf{h} \in \Omega_{(\lambda,c)}$, the line direction λ is contained in $\mathbf{G}_{\mathbb{R}}(1,\mathbf{h})$ the Grassmann space of line directions of the hyperplane $\mathbf{h}$.

The main result of the paper is the following

Theorem 6.2. *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a C^l definable function with $l \geqslant 2$. Assume that the function f satisfies condition* **SISI** *at c.*
If the function $t \mapsto |K|(t)$ is continuous at c, the function f is trivialised over a neighbourhood of c by means of the flow of a C^{l-1} definable vector field. So $c \notin B(f)$.

The rest of the section is devoted to the proof of this result.

To prove Theorem 6.2 we first need the following

Lemma 6.3. *Under the hypotheses of Theorem 6.2, $\tau_f(\mathscr{X}_c^{\infty})$ is of dimension at most $n-2$.*

Proof. From Proposition 3.3 and the continuity of the total absolute curvature function at c we deduce that $\cup_{\lambda \in X_c^{\infty}} \Omega_{(\lambda,c)}$ is of dimension at most $n-2$. ■

Proof. Let $X_c^{\infty,+}$ be the lift of X_c^{∞} onto $\mathbf{S}^{n-1}$. It is a closed definable subset of dimension at most $n-2$, let Λ_c^+ be the lift of Λ_c.

After a direct orthonormal change of coordinates if necessary, we can assume that the intersections of each coordinate axis with the unit sphere does meet $X_c^{\infty,+}$ and that any $u \in \Lambda_c^+$ does not lie in any coordinates hyperplane.

Let $\Delta := \{(\delta_1, \dots, \delta_n) : \forall i, \delta_i > 0\}$. Embedding Δ in $\mathrm{Gl}_n(\mathbb{R})$ as diagonal matrices makes Δ a smooth semi-algebraic subgroup with a smooth semi-algebraic action over $\mathbb{R}^n$. Note that Δ is diffeomorphic to $\mathbb{R}^n$.

For any $A \in \Delta$, let us define the following semi-algebraic function

$$g_A(x) = \langle A \cdot x, x \rangle^{\frac{1}{2}}.$$

Note that g_A is a smooth proper submersion outside the origin since such an A is positive definite.
We will recycle here the method used in [NZ] and [TZ] but with "spheres" given by the levels of a function g_A for an appropriate A.

For each $u \in \Lambda_c^+$, let $V(u)$ be the closed definable subset defined as

$$V(u) := \mathbf{clos}\{\nu \in \mathbf{S}^{n-1} : \nu = \lim \nu_f(x) \text{ with } |x| \to +\infty, \frac{x}{|x|} \to u \\ \text{and } f(x) \to c\}.$$

Obviously each $V(u)$, $u \in \Lambda_c^+$, is of dimension at most $n-2$.

Let $\Delta_0 := \{A \in \Delta : i \neq j \Longrightarrow \delta_i \neq \delta_j\}$. It is open, semi-algebraic and dense in Δ. Since in the new coordinate system any $u \in X_c^{\infty,+}$ has at least two non zero coordinates, no such vector u can be an eigenvector of $A \in \Delta_0$.
As a corollary of this fact we get

Lemma 6.4. *For each $A \in \Delta_0$, there exists $\alpha, \beta \in]0,1[$ such that there are $R > 0$ and $\varepsilon > 0$ such that for each $x \in f^{-1}(]c-\varepsilon, c+\varepsilon[) \setminus \mathbf{clos}(\mathbf{B}_R^n)$*

$$\alpha < \left\langle \frac{A \cdot x}{|A \cdot x|}, \frac{x}{|x|} \right\rangle < \beta.$$

Proof. Since $X_c^{\infty,+}$ is compact and by definition of Δ_0, there exists positive α_0 and β_0 such that for any $u \in X_c^{\infty,+}$,

$$\alpha_0 < \left\langle \frac{A \cdot u}{|A \cdot u|}, u \right\rangle < \beta_0 < 1.$$

Writing the definition of $X_c^{\infty,+}$ provides the desired statement. ∎

Given $u \in \Lambda_c^+$, each coordinates is non zero, so the subset $\Delta_0(u) := \{A \cdot u : A \in \Delta_0\}$ is a semi-algebraic open subset of $\mathbb{R}^n \setminus \{0\}$. So its image under the radial projection is open in $\mathbf{S}^{n-1}$. Let $\rho : \mathbb{R}^n \setminus \{0\} \mapsto \mathbf{S}^{n-1}$ be the radial projection. It is a smooth semi-algebraic map thus $U_{c \in \Lambda_c^+} \rho^{-1}(V(u))$ is a definable positive cone of dimension at most $n-1$. This means there exists Δ_1 an open dense definable subset of Δ such that for any $u \in \Lambda_c^+$ and for any $A \in \Delta_1$, $A \cdot u \notin V(u)$.
Given $u \in \mathbf{S}^{n-1}$ an $\eta > 0$, the positive conical neighbourhood of the oriented semi-line $\mathbb{R}^+ u$ of radius η is the following

$$C^+(u;\eta) := \{x \in \mathbb{R}^n \setminus \{0\} : |u - \rho(x)| < \eta\} \cup \{0\}.$$

Lemma 6.5. *Let $A \in \Delta_1$. There exists $\gamma, \delta \in]0,1[$ such that there are $R > 0$ and $\varepsilon > 0$, $\eta > 0$, such that for each $x \in (f^{-1}(]c-\varepsilon, c+\varepsilon[) \cap C^+(u;\eta)) \setminus \mathbf{clos}(\mathbf{B}_R^n)$,*

$$-1 < -\gamma < \left\langle \frac{A \cdot x}{|A \cdot x|}, \frac{\nabla f(x)}{|\nabla f(x)|} \right\rangle < \delta < 1.$$

Proof. From the definition of Δ_1 we deduce, there exists $\gamma_0, \delta_0 \in]0,1[$ such that for each $u \in \Lambda_c^+$, each $\nu \in V(u)$ and each $v \in \mathbf{S}^{n-1}$ such that $|v-u| < \eta_0$ for some positive η_0, we find

$$-1 < -\gamma_0 < \left\langle \frac{A \cdot v}{|A \cdot v|}, \nu \right\rangle < \delta_0 < 1.$$

Returning to the definitions of $X_c^{\infty,+}$ and $V(u)$ gives us the desired uniform version. ∎

We recall that $\nu_{g_A}(x) = \dfrac{\nabla g_A(x)}{|\nabla g_A(x)|} = \dfrac{A \cdot x}{|A \cdot x|}$. As a corollary of Lemma 6.4 and of Lemma 6.5 we obtain

Proposition 6.6. *For each $A \in \Delta_1$, there exists $\alpha, \beta, \gamma \in]0,1[$ such that there exist $R > 0$, $\varepsilon > 0$ and $\eta > 0$ such that for each $x \in f^{-1}(]c-\varepsilon, c+\varepsilon[) \setminus \mathbf{clos}(\mathbf{B}_R^n)$*
(i) $\alpha|x| < \langle \nu_{g_A}(x), x\rangle < \beta|x|$,
(ii) for each $u \in \Lambda_c^+$ such that $x \in C^+(u;\eta)$, $|\langle \nu_{g_A}(x), \nu_f(x)\rangle| < \gamma < 1$.

Let A be given as in Proposition 6.6. Assume that R and ε are given. Let us define the following vector field

$$\omega_A(x) := \nu_f(x) - \langle \nu_f(x), \nu_{g_A}(x)\rangle \nu_{g_A}(x).$$

For $x \in f^{-1}(]c-\varepsilon, c+\varepsilon[) \setminus \mathbf{clos}(\mathbf{B}_R^n)$ this vector field is non vanishing, since $|\omega_A(x)| \geqslant \sqrt{1-\gamma^2}$, and is tangent to the levels of g_A (which are compact) and is transverse to the levels of f.

Let δ be the biggest eigenvalue of A. Now we define the vector field that will realise the trivialisation around c:

For x such that $g_A(x) \leqslant \delta R$, let $\xi(x) := \nu_f(x)$,

for x such that $g_A(x) \geqslant 2\delta R$, let $\xi(x) := \dfrac{\omega_A(x)}{|\omega_A(x)|}$,

for $g_A^{-1}(x) \in [\delta R, 2\delta R]$, let

$$\xi(x) := \kappa(g_A(x))\nu_f(x) + [1 - \kappa(g_A(x))]\frac{\omega_A(x)}{|\omega_A(x)|},$$

where $\kappa : [\delta R, 2\delta R] \mapsto [0,1]$ is a C^l definable function that is strictly decreasing and such that $\kappa(\delta R) = 1$ and $\kappa(2\delta R) = 0$, and is also l-flat at δR and at $2\delta R$.

Restricting ξ to $f^{-1}(]c-\varepsilon, c+\varepsilon[)$, we observe that $\xi(x)$ does not vanish and so the trivialisation of f in a neighbourhood of c is provided by the flow of

ξ as in [NZ] and [LZ]. ∎

7. Real polynomials versus complex polynomials

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $f : \mathbb{K}^n \mapsto \mathbb{K}$ be a polynomial of degree $d \geqslant 2$. Let ∇f be the polynomial vector field $\sum_i \partial_{x_i} f \partial_{x_i}$.

If $|\nabla f(x)| \to 0$ along a sequence x, such that $|x| \to +\infty$ and $x/|x| \to \mathbf{u} \in \mathbb{P}^{n-1}_{\mathbb{K}}$, we deduce that for each $i = 1, \ldots, n$, $\partial_{x_i} f_d(\mathbf{u}) = 0$, when $f = f_d + f_{d-1} + \ldots + f(0)$ is written as the sum of its homogeneous components. Assume moreover that along this sequence the Malgrange condition fails at c a regular value, that is $|x| \cdot |\nabla f(x)| \to 0$ and $f(x) \to c$.

Assume, after a rotation, that $\mathbf{u} = (0, \ldots, 0, 1)$, then writing $y_i = x_i/x_n$ and $y_0 = 1/x_n$, we deduce that along the sequence x

$$|(\partial_{y_1} \tilde{f}^{(n)}, \ldots, \partial_{y_{n-1}} \tilde{f}^{(n)}, d\tilde{f}^{(n)} - y_0 \partial_{y_0} \tilde{f}^{(n)})| \ll |y_0|^d,$$

with

$$\tilde{f}^{(n)}(y_0, \ldots, y_{n-1}) = y_0^d f(y_1/y_0, \ldots, y_{n-1}/y_0, 1/y_0).$$

We deduce that for $i = 1, \ldots, n-1$, $\partial_{y_i} f_d(\mathbf{u}) = 0$, and so $\partial_{x_n} f_d(\mathbf{u}) = 0$. So we also find that $|\partial_{y_0} \tilde{f}^{(n)}| \leqslant const \cdot |y_0|^{d-1}$, thus $f_{d-1}(0, \ldots, 0, 1) = f_{d-1}(\mathbf{u}) = 0$.

These elementary computations mean that the set of points $\mathbf{u}$ at infinity nearby which Malgrange condition fails at c is very specific, namely these points are roots of f_{d-1} and of ∇f_d.

Let $\lambda \in \mathbf{G}_{\mathbb{K}}(1, n)$ be a line direction and let $\mathbf{h}_\lambda \in \mathbf{G}_{\mathbb{K}}(n-1, n)$ be the hyperplane direction orthogonal to λ. We denote by $\mathbf{P}_\lambda(f)$ the subset $\{x \notin \mathbf{crit}(f) : T_x f = \mathbf{h}_\lambda\}$, that is the polar variety of the function f in the line direction λ. Note that it is a semi-algebraic subset of $\mathbb{K}^n$.

Lemma 7.1. *Let c be a regular value of f. Assume there exists a line direction λ such that there exists a sequence (x) in $\mathbf{P}_\lambda(f)$, $|x| \to +\infty$, $x/|x| \to \mathbf{u}$ and $f(x) \to c$. If $f_{d-1}(\mathbf{u}) \neq 0$, then $\nabla f_d(\mathbf{u}) \neq 0$ and so $\lambda = \mathbb{R}\nu_{f_d}(\mathbf{u})$.*

Proof. We can assume that this phenomenon occurs along a semi-algebraic path, say $\Gamma \subset \mathbf{P}_\lambda(f)$ and is parametrised as $]0, \varepsilon[\ni r \mapsto x(r)$ such that $|x(r)| \to +\infty$ as $r \to 0$ and $r|x(r)| \to 1$.
Assume that $f_{d-1}(\mathbf{u}) = \alpha_0 \neq 0$. Thus there exists a positive constant M such that for r small enough

$$|x(r)| \cdot |\nabla f(x(r))| \geqslant M, \text{ that is } |\nabla f(x(r))| \geqslant M \cdot r.$$

Since along the path $|\partial_r f| \sim |x| \cdot |f - c|$, we deduce that $|\nabla' f| \gg |x| \cdot |f - c|$. From Lemma 5.6 we get $\langle \mathbf{u}, \lambda \rangle = 0$.

After an orthonormal change of coordinates we assume that $\mathbf{u} = (0, \ldots, 0, 1)$. Writing $\lambda = (\lambda_1, \ldots, \lambda_n)$ and since $\langle \mathbf{u}, \lambda \rangle = 0$, after a rotation in $\lambda_1, \ldots, \lambda_{n-1}$

we actually get $\lambda_1 = \pm 1$ and $\lambda_2 = \ldots = \lambda_n = 0$. So we get $(\partial_{x_i} f)_{|\Gamma} = 0$, $i = 2, \ldots, n$.

Let $y_i := x_i/x_n$, for $i = 1, \ldots, n-1$, and $y_0 := 1/x_n$. Thus we get

$$\tilde{f}^{(n)}(y_0, \ldots, y_{n-1}) := y_0^d f(y_1/y_0, \ldots, y_{n-1}/y_0, 1/y_0).$$

By abuse of notation let us define

$$\partial_{y_n} \tilde{f}^{(n)}(y_0, \ldots, y_{n-1}) := y_0^{d-1} \partial_{x_n} f(y_1/y_0, \ldots, y_{n-1}/y_0, 1/y_0).$$

Thus we get $d \cdot \tilde{f}^{(n)} = \partial_{y_n} \tilde{f}^{(n)} + \sum_{i=0}^{n-1} y_i \partial_{y_i} \tilde{f}^{(n)}$, and so

$$|y_0|^{d-1} \cdot |\nabla f(x)| = |(\partial_{y_1} \tilde{f}^{(n)}(y), \ldots, \partial_{y_n} \tilde{f}^{(n)}(y))|.$$

Let $y(r) = (y_0(r), \ldots, y_{n-1}(r))$ be the path in the new coordinates. Thus we get $y_0 \simeq r$ and $r^{-1} y_i(r) \to 0$ as r goes to 0. Thus we can assume $y_0 = r$. Thus along Γ we know that $r^{-d} \cdot |\tilde{f}^{(n)}(y(r)) - cr^d| \to 0$ as r goes to 0, and

$$|(\partial_{y_1} \tilde{f}^{(n)}(y), \ldots, \partial_{y_n} \tilde{f}^{(n)}(y))| \geqslant Mr^d, \text{ that is } |\partial_{y_1} \tilde{f}^{(n)}(y)| \geqslant Mr^d.$$

Along Γ we deduce that

$$\begin{aligned} d \cdot \tilde{f}^{(n)}(y(r)) = r\partial_{y_0} \tilde{f}^{(n)}(y(r)) + y_1 \partial_{y_1} \tilde{f}^{(n)}(y(r)) = \\ r f_{d-1}(y(r)) + y_1(r) \partial_{y_1} f_d(y(r)) + o(r). \end{aligned}$$

Taking the derivative of $\tilde{f}^{(n)}(y(r))$ in r provides

$$\partial_{y_0} \tilde{f}^{(n)} + y_1' \partial_{y_1} \tilde{f}^{(n)} \simeq dcr^{d-1}.$$

So if $\partial_{y_1} \tilde{f}^{(n)} \simeq \alpha r^a$ and $y_1(r) \simeq \beta r^b$ and since $d \geqslant 2$ we deduce that $a+b = 1$ and $\alpha_0 + b\alpha\beta = 0$. But we also deduce that $\alpha_0 + \beta\alpha = 0$ so $b = 1$ and thus $a = 0$. Which implies that $\partial_{y_1} f_d(\mathbf{u}) = \alpha \neq 0$ and so the claim is proved. ■

Let us recall what is happening in the complex case. Parusiński defined in [Pa1] the notion of complex polynomial with isolated singularities at infinity. By this he means that the subset

$$A := \{\lambda \in \mathbb{H}_{\mathbb{C}}^{\infty} : \partial_{x_1} f_d(\lambda) = \ldots = \partial_{x_n} f_d(\lambda) = f_{d-1}(\lambda) = 0\},$$

is finite. For each $t \in \mathbb{C}$, let $\mathbf{F}_t \subset \mathbb{P}_{\mathbb{C}}^n$ be the projective closure of the level $f^{-1}(t)$. So for each regular value t, the projective hypersurface $\mathbf{F}_t$ has only isolated singularities. Let $\mu(\mathbf{F}_t)$ be the sum of the Milnor numbers of the isolated singularities of $\mathbf{F}_t$.

The notion of relative conormal bundle and relative conormal space, defined in Section 4, still make sense for a complex polynomial when dealing with the complex analogs of the notions used in the real setting. Note then, once $t \notin K_\infty(f)$, we find $X_t = \mathbf{F}_t$.

For a given line direction v, let $\mathbf{clos}(\mathbf{P}_\lambda(f))$ be the projective closure of $\mathbf{P}_\lambda(f) \subset \mathbb{C}^n \setminus \mathbf{crit}(f) \subset \mathbb{P}_{\mathbb{C}}^n$. Then

Theorem 7.2 ([Pa1, Ti])**.** *Let c be a regular value of the complex polynomial f with isolated singularities at infinity. The following statements are equivalent:*

(1) $c \notin K_\infty(f)$.
(2) $c \notin B(f)$.
(3) The total Milnor number function $t \mapsto \mu(\mathbf{F}_t)$ is locally constant in a neighbourhood of c.
(4) The Euler Characteristic function $t \mapsto \chi(\mathbf{F}_t)$ is locally constant in a neighbourhood of c.
(5) For λ in a Zariski open set of $\mathbf{G}_\mathbb{C}(1,n)$, $\mathbf{clos}(\mathbf{P}_\lambda(f)) \cap X_c^\infty = \emptyset$.
(6) The dimension of $\mathscr{X}_c^\infty$ is at most $n-2$.

As far as the author knows, there is no real polynomial version of such a statement. One of the reason for this is that the local constancy of simple invariants such as the Euler Characteristic or some Milnor numbers is not usually a sufficient condition to ensure the equisingularity of a family.

Let us mention the connection between Theorem 7.2 and the complex version of Theorem 5.1, which holds true with $\theta_c(t) = K_c t^{\rho_c}$ for a rational number $\rho_c \leqslant 1$ and $K_c > 0$.

Proposition 7.3 ([DG2])**.** *Let f be a complex polynomial with isolated singularities at infinity. Let c be a regular value. The exponent ρ_c is equal to* 1 *if and only if $c \in B(f)$.*

The real version of this result is not true, as given by $f(x,y) = -y(2x^2y^2 - 9xy + 12)$ (see [DG1, DG2] for more on this example).

For real polynomial functions we have the following

Proposition 7.4. *Let f be a real polynomial function on $\mathbb{R}^n$ of degree d. Assume the following subset*

$$A := \{\lambda \in \mathbb{H}_\mathbb{R}^\infty : \partial_{x_1} f_d(\lambda) = \ldots = \partial_{x_n} f_d(\lambda) = f_{d-1}(\lambda) = 0\}$$

is finite. Given any sequence x such that $|x| \to +\infty$, $f(x) \to c \notin K_0(f)$ and $x/|x| \to \lambda \notin A$, any limit of tangent hyperplane direction $\mathbf{h} = \lim T_x F$ contains the line direction λ, so the function f satisfies condition **SISI** *at c.*

Proof. Let $r \mapsto ru(r)$ be a semi-algebraic path such that $|u(r)| = 1$ and $f(ru(r)) \to c$ as $r \to +\infty$. Assume that $u(r) \to \mathbf{u} \in \mathbf{S}^{n-1}$ as $r \to +\infty$ such that $f_{d-1}(\mathbf{u}) = \alpha_0 \neq 0$. We must either have $f_d(\mathbf{u}) = 0$ or $\langle \nu_{f_d}(\mathbf{u}), \mathbf{u}\rangle = 0$. After a rotation we assume that $\mathbf{u} = (0,\ldots,0,1)$. Writing $u(r) = (u_1(r), u_2(r), \ldots, 1 - u_n(r))$, we find that $u_i(r) \sim r^{-e_i}$ for rational positive numbers e_i, $i = 1,\ldots,n$. Let $e = \min\{e_i\}$. Then we find that $e_n = 2e$. For each $i = 1,\ldots n$, we get $\partial_{x_i} f_d(u(r)) \sim r^{-d_i}$, for some positive rational numbers d_i. Since $f(ru(r)) \to c$, we must have $|rf_d(u(r)) + f_{d-1}(u(r))| \leqslant const \cdot r^{-1}$. So $2e \leqslant 1$ and $f_d(u(r)) \simeq -\alpha_0 r^{-1}$. We deduce that

$$\langle \nabla f_d(u(r)), u(r)\rangle \simeq -d\alpha_0 r^{-1} \text{ and } \langle \nabla f_d(u(r)), u'(r)\rangle \simeq d\alpha_0 r^{-2}.$$

If $d_n \geqslant 1$, then

$$\langle \nabla f_d(u(r)), u'(r)\rangle \simeq u_2'(r)\partial_{x_2} f_d(u(r)) + \ldots + u_n'(r)\partial_{x_n} f_d(u(r)) \simeq d\alpha_0 r^{-2}.$$

There exists $i \in \{2,\ldots,n\}$ such that $e_i + d_i \leqslant 1$. So

$$\partial_{x_i} f_d(u(r)) + r^{-1}\partial_{x_i} f_{d-1}(u(r)) \sim r^{-d_i}$$

thus $\nabla f(ru(r)) \simeq \nabla f_d(ru(r))$ and moreover $\langle \nabla f_d/|\nabla f_d|, \mathbf{u}\rangle \to 0$.
If $d_n < 1$, there exists $i \in \{2,\ldots,n\}$ such that $e_i + d_i \leqslant d_n$ and thus again we deduce $\nabla f(ru(r)) \simeq \nabla f_d(ru(r))$ and $\langle \nabla f_d/|\nabla f_d|, \mathbf{u}\rangle \to 0$. This ends the proof. ∎

So condition **SISI** is a reasonable condition to work with in the frame we are given. The finiteness of the subset A has also another consequence. As a corollary we obtain

Proposition 7.5. *Let f be as in Proposition 7.4.*
(1) If $c \notin K(f)$, then the function $t \mapsto |K|(t)$ is continuous at c.
(2) If the total curvature function $|K|$ is not continuous at a regular value c, then $c \in K_\infty(f)$ and moreover its exponent at infinity is 1.

Proof. Point (1) comes from the proof of Lemma 7.1.
For point (2), the discontinuity of the total curvature function at c guarantees we can apply Lemma 7.1. So there exists $v \in \mathbf{S}^{n-1}$ such that along the oriented polar curve in the oriented direction v, namely $\Psi_f^{-1}(\{v\} \times \mathbb{R})$ is not empty and moreover there exists a branch Γ_v of this polar curve such that $f_{|\Gamma_v}(x) \to c$ as $|x| \to +\infty$ along which $|(\nabla f)_{|\Gamma_v}|(x) \sim |\partial_r f_{|\Gamma_v}|(x)$, that is $\rho_c = 1$. ∎

This partially answers, in the real polynomial case, the question about the values at which the total absolute curvature is not continuous: When regular they can only be asymptotic critical values c with $\rho_c = 1$.

As a final corollary we deduce

Corollary 7.6. *Let f be a real polynomial function on $\mathbb{R}^n$ of degree d such that*

$$A := \{\lambda \in \mathbb{H}^\infty_{\mathbb{R}} : \partial_{x_1} f_d(\lambda) = \ldots = \partial_{x_n} f_d(\lambda) = f_{d-1}(\lambda) = 0\}$$

is finite. If c is a regular bifurcation value of f, then the function total curvature of f is not continuous at c, or equivalently there exists a non empty open subset $\Lambda \subset \mathbf{G}_{\mathbb{R}}(1,n)$ such that for each $\lambda \in \Lambda$, $\mathbf{P}_\lambda(f)$ is a non empty smooth curve and $\mathbf{clos}(\mathbf{P}_\lambda(f)) \cap X_c^\infty \cap A \neq \emptyset$.

8. Comments and remarks

The reader will have noticed that the conclusion of Theorem 6.2 still holds true if we drop the hypothesis on the continuity of the total absolute curvature function, to only requiring that $\tau_f(\mathscr{X}_c^\infty)$ - the closure of the limits

of the tangent spaces to the fibres as they tend to c - is a proper closed subset of $\mathbf{G}_{\mathbb{R}}(n-1,n)$. The proof works the same. But as noticed by Tibăr in the real polynomial case [Ti], if $\tau_f(\mathscr{X}_c^\infty)$ is of dimension $n-1$ then it is $\mathbf{G}_{\mathbb{R}}(n-1,n)$. In this situation finer conditions (yet unknown) will be required to ensure the trivialisation, even in the case of **SISI**.

We have said that Theorem 6.2 was the real counterpart of Theorem 7.2. Our result only provides a sufficient condition expressed in terms of total absolute curvature (or in terms of polar curves) once specified the points at infinity (that is in $\mathbb{H}_{\mathbb{R}}^\infty$) at which the Gauss-Kronecker curvature may concentrate.

In the complex domain, to each $t \in \mathbb{C}$, we associate a real number $LK(t)$, which is just the total $(2n-2)$-Lipschitz-Killing curvature of the regular part of the real algebraic subset $F_t \subset \mathbb{C}^n$ (see [TS] for how this can be used in equisingularity problems). It is of constant sign and it is obvious that point (vi) of Theorem 7.2 is equivalent to the continuity at c of the function $t \mapsto LK(t)$. So in a sense, since it is equivalent to point (iv), Theorem 7.2 can be also considered as a Gauss-Bonnet-Chern type constancy result.

Having isolated singularities at infinity in the complex case is almost exactly requiring which points at infinity are likely to concentrate curvature. The finiteness of $\{\partial_{x_1} f_d = \ldots = \partial_{x_n} f_1 = f_{d-1} = 0\} \cap \mathbb{H}_{\mathbb{C}}^\infty$ implies that if there are generic polar curves along a non bounded branch of which f tends to a regular value c, this branch must tend to a point in $\{\partial_{x_1} f_d = \ldots = \partial_{x_n} f_1 = f_{d-1} = 0\} \cap \mathbb{H}_{\mathbb{C}}^\infty$. Thus, the knowledge of the set of points at infinity where the generic polar curves (or the "generic" non-empty polar varieties when the Gauss map is degenerate) are ending seems to be an interesting object to understand and to describe when we are willing to decide whether a regular value is a bifurcation value or not, see for instance [Ti, Example 2.13].

In the real domain, the polynomial case is already delicate since point (i) and point (ii) of Theorem 7.2 are already not equivalent (see [TZ]). Moreover even among regular asymptotic critical values there are distinctions to make as suggested in Section 5.

King-Zaharia-Tibăr example $f(x,y) = -y(2x^2y^2 - 9xy + 12)$ in the real plane, which satisfies our hypotheses, has no bifurcation value, but $0 \in K_\infty(f)$ and $\lim_{t\to 0} |K(t)| = 2\pi$ while $|K(0)| = 0$. The trivialisation cannot be realised by any flow of a vector field tangent to ∇f.

From [DG1], in the real plane case, we deduce that for a regular value c has its exponent $\rho_c = 1$ if and only if the total curvature function $t \mapsto |K|(t)$ is not continuous at c.

For real polynomial with isolated singularity at infinity is $\rho_c < 1$ equivalent to the continuity at c of $t \mapsto |K|(t)$?

More generally, we wonder if having a degenerate Gauss map, that is of rank at most $n-2$, is compatible with condition **SISI**. I really doubt it for polynomials.

Another way to say that is to ask, in the affine domain as well as at infinity, what conditions on the singularities of the (generalised) critical levels (when looking at their closure in $\mathbb{P}^n_{\mathbb{R}}$) of the function, a degenerate Gauss map is carrying ?

Thanks

The author had been partially supported by the *European research network IHP-RAAG* contract number HPRN-CT-2001-00271, and by *Deutsche Forschungs-Gemeinschaft* in the Priority Program *Global Differential Geometry.*
We would like to thank the University of Bath (UK) and Carl von Ossietzky Universität Oldenburg (Germany) for the working conditions provided while working on this paper.
Last, but not least, a big thank you to Nicolas Dutertre for talks, questions, comments, and for a very careful reading that helped a lot. Cheers Mate !

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